

HOMOTOPICALLY EQUIVALENT SIMPLE LOOPS ON 2-BRIDGE SPHERES IN HECKOID ORBIFOLDS FOR 2-BRIDGE LINKS (I)

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ABSTRACT. In this paper and its sequel, we give a necessary and sufficient condition for two essential simple loops on a 2-bridge sphere in an even Heckoid orbifold for a 2-bridge link to be homotopic in the orbifold. We also give a necessary and sufficient condition for an essential simple loop on a 2-bridge sphere in an even Heckoid orbifold for a 2-bridge link to be peripheral or torsion in the orbifold. This paper treats the case when the 2-bridge link is a $(2, p)$ -torus link, and its sequel will treat the remaining cases.

1. INTRODUCTION

Let $K(r)$ be the 2-bridge link of slope $r \in \mathbb{Q}$ and let n be an integer or a half-integer greater than 1. In [9], following Riley's work [19], we introduced the *Heckoid group* $G(r; n)$ of index n for $K(r)$ as the orbifold fundamental group of the *Heckoid orbifold* $\mathcal{S}(r; n)$ of index n for $K(r)$. According to whether n is an integer or a non-integral half-integer, the Heckoid group $G(r; n)$ and the Heckoid orbifold $\mathcal{S}(r; n)$ are said to be *even* or *odd*. The even Heckoid orbifold $\mathcal{S}(r; n)$ is the 3-orbifold such that

- (i) the underlying space $|\mathcal{S}(r; n)|$ is the exterior, $E(K(r)) = S^3 - \text{int } N(K(r))$, of $K(r)$, and
- (ii) the singular set is the lower tunnel of $K(r)$, where the index of the singularity is n .

For a description of odd Heckoid orbifolds, see [9, Proposition 5.3].

In [9, Theorem 2.3], we gave a systematic construction of upper-meridian-pair-preserving epimorphisms from 2-bridge link groups onto Heckoid groups,

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generalizing Riley's construction in [19]. Furthermore, we proved, in [10, Theorem 2.4], that all upper-meridian-pair-preserving epimorphisms from 2-bridge link groups onto *even* Heckoid groups are contained in those constructed in [9, Theorem 2.3]. To prove this result, we determined those essential simple loops on a 2-bridge sphere in an even Heckoid orbifold $\mathbf{S}(r; n)$ which are null-homotopic in $\mathbf{S}(r; n)$ (see [10, Theorem 2.3]).

The purpose of this paper and its sequel [14] is (i) to give a necessary and sufficient condition for two essential simple loops on a 2-bridge sphere in an even Heckoid orbifold $\mathbf{S}(r; n)$ to be homotopic in $\mathbf{S}(r; n)$, and (ii) to give a necessary and sufficient condition for an essential simple loop on a 2-bridge sphere in an even Heckoid orbifold $\mathbf{S}(r; n)$ to be peripheral or torsion in $\mathbf{S}(r; n)$. In this paper, we treat the case when $K(r)$ is a $(2, p)$ -torus link, and the sequel [14] will treat the remaining cases. These results will be used, in our upcoming work, to show the existence of a variation of McShane's identity for even Heckoid orbifolds. For an overview of this series of works, we refer the reader to the research announcement [7].

The remainder of this paper is organized as follows. In Section 2, we describe the main results. In Section 3, we introduce the upper presentation of an even Heckoid group, and review basic facts concerning its single relator established in [6]. In Section 4, we apply small cancellation theory to conjugacy diagrams over the upper presentations of even Heckoid groups. In Section 5, we establish technical lemmas which will play essential roles in the succeeding sections. Finally, Sections 6–8 are devoted to the proof of Main Theorem 2.5.

2. MAIN RESULTS

We quickly recall notation and basic facts introduced in [9]. The *Conway sphere* \mathbf{S} is the 4-times punctured sphere which is obtained as the quotient of $\mathbb{R}^2 - \mathbb{Z}^2$ by the group generated by the π -rotations around the points in \mathbb{Z}^2 . For each $s \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let α_s be the simple loop in \mathbf{S} obtained as the projection of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope s . We call s the *slope* of the simple loop α_s .

For each $r \in \hat{\mathbb{Q}}$, the *2-bridge link* $K(r)$ of slope r is the sum of the rational tangle $(B^3, t(\infty))$ of slope ∞ and the rational tangle $(B^3, t(r))$ of slope r . Recall that $\partial(B^3 - t(\infty))$ and $\partial(B^3 - t(r))$ are identified with \mathbf{S} so that α_∞ and α_r bound disks in $B^3 - t(\infty)$ and $B^3 - t(r)$, respectively. By van-Kampen's theorem, the link group $G(K(r)) = \pi_1(S^3 - K(r))$ is obtained as follows:

$$G(K(r)) = \pi_1(S^3 - K(r)) \cong \pi_1(\mathbf{S}) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong \pi_1(B^3 - t(\infty)) / \langle\langle \alpha_r \rangle\rangle.$$

We call the image in the link group of the meridian pair of $\pi_1(B^3 - t(\infty))$ the *upper meridian pair* (see [6, Figure 3]).

On the other hand, if r is a rational number and $n \geq 2$ is an integer, then by the description of the even Heckoid orbifold $\mathbf{S}(r; n)$ in the introduction, the even Hekoid group $G(r; n) = \pi_1(\mathbf{S}(r; n))$ is identified with

$$G(r; n) \cong \pi_1(\mathbf{S}) / \langle \langle \alpha_\infty, \alpha_r^n \rangle \rangle \cong \pi_1(B^3 - t(\infty)) / \langle \langle \alpha_r^n \rangle \rangle.$$

In particular, the even Heckoid group $G(r; n)$ is a two-generator and one-relator group. We call the image in $G(r; n)$ of the meridian pair of $\pi_1(B^3 - t(\infty))$ the *upper meridian pair*.

We are interested in the following naturally arising question.

Question 2.1. For r a rational number and n an integer or a half-integer greater than 1, consider the Heckoid orbifold $\mathbf{S}(r; n)$ of index n for the 2-bridge link $K(r)$.

- (1) Which essential simple loop α_s on \mathbf{S} is null-homotopic in $\mathbf{S}(r; n)$?
- (2) For two distinct essential simple loops α_s and $\alpha_{s'}$ on \mathbf{S} , when are they homotopic in $\mathbf{S}(r; n)$?
- (3) Which essential simple loop α_s on \mathbf{S} is peripheral or torsion in $\mathbf{S}(r; n)$?

This is an analogy of a natural question for 2-bridge links, which has origin in Minsky's question [3, Question 5.4], and which was completely solved in the series of papers [6, 11, 12, 13] and applied in [8]. See [5] for an overview of these works and [16] for a related work.

We note that (1) a loop in the orbifold $\mathbf{S}(r; n)$ is *null-homotopic* in $\mathbf{S}(r; n)$ if and only if it determines the trivial conjugacy class of the Heckoid group $G(r; n)$, and (2) two loops in $\mathbf{S}(r; n)$ are *homotopic* if and only if they determine the same conjugacy class in $G(r; n)$ (see [1, 2] for the concept of homotopy in orbifolds). We say that a loop in $\mathbf{S}(r; n)$ is *peripheral* if and only if it is homotopic to a loop in the paring annulus naturally associated with $\mathbf{S}(r; n)$ (see [9, Section 6]), i.e., it represents the conjugacy class of a power of a meridian. We say that a loop in $\mathbf{S}(r; n)$ is *torsion* if it represents the conjugacy class of a non-trivial torsion element of $G(r; n)$. If we identify $G(r; n)$ with a Kleinian group generated by two parabolic transformations (see [9, Theorem 2.2]), then a loop $\mathbf{S}(r; n)$ is peripheral or torsion if and only if it corresponds to a parabolic transformation or a non-trivial elliptic transformation accordingly. Thus Question 2.1 can be interpreted as a question on the Heckoid group $G(r; n)$.

Let \mathcal{D} be the *Farey tessellation* of the upper half plane \mathbb{H}^2 . Then $\hat{\mathbb{Q}}$ is identified with the set of the ideal vertices of \mathcal{D} . Let Γ_∞ be the group of automorphisms of \mathcal{D} generated by reflections in the edges of \mathcal{D} with an endpoint

∞ . For r a rational number and n an integer or a half-integer greater than 1, let $C_r(2n)$ be the group of automorphisms of \mathcal{D} generated by the parabolic transformation, centered on the vertex r , by $2n$ units in the clockwise direction, and let $\Gamma(r; n)$ be the group generated by Γ_∞ and $C_r(2n)$. Suppose that r is not an integer, i.e., $K(r)$ is not a trivial knot. Then $\Gamma(r; n)$ is the free product $\Gamma_\infty * C_r(2n)$ having a fundamental domain, R , shown in Figure 1. Here, R is obtained as the intersection of fundamental domains for Γ_∞ and $C_r(2n)$, and so R is bounded by the following two pairs of Farey edges:

- (i) the pair of adjacent Farey edges with an endpoint ∞ which cuts off a region in \mathbb{H}^2 containing r , and
- (ii) a pair of Farey edges with an endpoint r which cuts off a region in \mathbb{H}^2 containing ∞ such that one edge is the image of the other by a generator of $C_r(2n)$.

Let $\bar{I}(r; n)$ be the union of two closed intervals in $\partial\mathbb{H}^2 = \hat{\mathbb{R}}$ obtained as the intersection of the closure of R and $\partial\mathbb{H}^2$. (In the special case when $r \equiv \pm 1/p \pmod{\mathbb{Z}}$ for some integer $p > 1$, one of the intervals may be degenerated to a single point.) Note that there is a pair $\{r_1, r_2\}$ of boundary points of $\bar{I}(r; n)$ such that r_2 is the image of r_1 by a generator of $C_r(2n)$. Set $I(r; n) := \bar{I}(r; n) - \{r_i\}$ with $i = 1$ or 2 . Note that $I(r; n)$ is the disjoint union of a closed interval and a half-open interval, except possibly for the special case when $r \equiv \pm 1/p \pmod{\mathbb{Z}}$. Even in the exceptional case, we can choose R so that $I(r; n)$ satisfies this condition.

As a sufficient condition for each of Question 2.1(1) and (2), we were able to obtain the following (cf. [9, Theorem 2.4]).

Proposition 2.2 ([10, Theorem 2.2]). *Suppose that r is a non-integral rational number and that n is an integer or a half-integer greater than 1. Then, for any $s \in \hat{\mathbb{Q}}$, there is a unique rational number $s_0 \in I(r; n) \cup \{\infty, r\}$ such that s is contained in the $\Gamma(r; n)$ -orbit of s_0 . Moreover the conjugacy classes α_s and α_{s_0} in $G(r; n)$ are equal. In particular, if $s_0 = \infty$, then α_s is the trivial conjugacy class in $G(r; n)$.*

Furthermore, for even Heckoid groups, we proved that the converse of the last assertion of Proposition 2.2 is also true, implying that the sufficient condition for Question 2.1(1) is actually a necessary and sufficient condition. This made us possible to describe all upper-meridian-pair-preserving epimorphisms from 2-bridge link groups onto even Heckoid groups.

Proposition 2.3 ([10, Theorem 2.3]). *Suppose that r is a non-integral rational number and that n is an integer greater than 1. Then α_s represents the trivial*

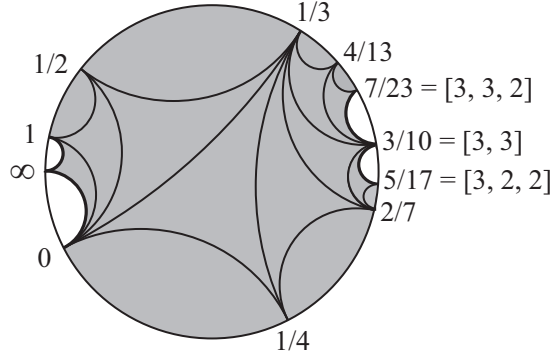


FIGURE 1. A fundamental domain of $\Gamma(r; n)$ in the Farey tessellation (the shaded domain) for $r = 3/10 = \frac{1}{3 + \frac{1}{3}} =: [3, 3]$ and $n = 2$. In this case, $\bar{I}(r; n) = [0, 5/17] \cup [7/23, 1]$.

element of $G(r; n)$ if and only if s belongs to the $\Gamma(r; n)$ -orbit of ∞ . In other words, if $s \in I(r; n) \cup \{r\}$, then α_s does not represent the trivial element of $G(r; n)$.

Remark 2.4. When r is an integer, the Heckoid group $G(r; n) \cong G(0; n)$ is isomorphic to the subgroup $\langle P, SPS^{-1} \rangle$ of the classical *Hecke group* $\langle P, S \rangle$ introduced in [4], where

$$P = \begin{pmatrix} 1 & 2 \cos \frac{\pi}{2n} \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Moreover, the group $\Gamma(0; n)$ is the free product of three cyclic groups of order 2 generated by the reflections in the Farey edges $\langle \infty, 0 \rangle$ and $\langle \infty, 1 \rangle$ and the geodesic $1, 1/n$. (The last geodesic is a Farey edge if n is an integer, whereas it bisects a pair of adjacent Farey triangles if n is a non-integral half-integer.) The region of \mathbb{H}^2 bounded by these three geodesics is a fundamental domain for the action of $\Gamma(0; n)$ on \mathbb{H}^2 . It is easy to see that Proposition 2.2 continues to be valid when r is an integer, provided that we set $I(0; n) := [1/n, n]$. It is plausible that Proposition 2.3 is also valid even when r is an integer. However, we cannot directly apply the arguments of [10], and this case will be treated elsewhere.

The purpose of the present paper and its sequel [14] is to give a complete solution to each of Question 2.1(2) and (3) for even Heckoid groups. In this

paper, we give an answer to Question 2.1(2) and (3) when $K(r)$ is a $(2, p)$ -torus link.

Main Theorem 2.5. *Suppose that p and n are integers greater than 1. Then the following hold.*

- (1) *The simple loops $\{\alpha_s \mid s \in I(1/p; n)\}$ represent mutually distinct conjugacy classes in $G(1/p; n)$.*
- (2) *There is no rational number $s \in I(1/p; n)$ for which α_s is peripheral in $G(1/p; n)$.*
- (3) *There is no rational number $s \in I(1/p; n)$ for which α_s is torsion in $G(1/p; n)$.*

Note that (1) together with (3) implies that the simple loops $\{\alpha_s \mid s \in I(1/p; n) \cup \{1/p\}\}$ are not mutually homotopic in $\mathbf{S}(1/p; n)$, since $\alpha_{1/p}$ is clearly torsion in $\mathbf{S}(1/p; n)$.

As in [10], the key tool used in the proofs is small cancellation theory, applied to two-generator and one-relator presentations, so-called the upper presentations, of even Heckoid groups.

3. UPPER PRESENTATIONS OF EVEN HECKOID GROUPS AND REVIEW OF BASIC FACTS FROM [6]

In this section, we introduce the upper presentation of even Heckoid groups $G(r; n)$, where r is a rational number and n is an integer with $n \geq 2$, and review basic facts established in [6] concerning it. These facts are to be used throughout this paper and its sequel [7].

In order to describe the upper presentations of even Heckoid groups, recall that

$$G(r; n) \cong \pi_1(\mathbf{S}) / \langle\langle \alpha_\infty, \alpha_r^n \rangle\rangle \cong \pi_1(B^3 - t(\infty)) / \langle\langle \alpha_r^n \rangle\rangle.$$

Let $\{a, b\}$ be the standard meridian generator pair of $\pi_1(B^3 - t(\infty), x_0)$ as described in [6, Section 3] (see also [5, Section 5]). Then $\pi_1(B^3 - t(\infty))$ is identified with the free group $F(a, b)$. Obtain a word $u_r \in F(a, b) \cong \pi_1(B^3 - t(\infty))$ which is represented by the simple loop α_r . It then follows that

$$G(r; n) \cong \pi_1(B^3 - t(\infty)) / \langle\langle \alpha_r^n \rangle\rangle \cong \langle a, b \mid u_r^n \rangle.$$

This two-generator and one-relator presentation is called the *upper presentation* of the even Heckoid group $G(r; n)$, which is used throughout the remainder of this paper. It is known by [18, Proposition 1] that there is a nice formula to find u_r as follows. (For a geometric description, see [5, Section 5].)

Lemma 3.1. Let p and q be relatively prime integers such that $p \geq 1$. For $1 \leq i \leq p-1$, let

$$\epsilon_i = (-1)^{\lfloor iq/p \rfloor},$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x .

(1) If p is odd, then

$$u_{q/p} = a \hat{u}_{q/p} b^{(-1)^q} \hat{u}_{q/p}^{-1},$$

$$\text{where } \hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \dots b^{\epsilon_{p-2}} a^{\epsilon_{p-1}}.$$

(2) If p is even, then

$$u_{q/p} = a \hat{u}_{q/p} a^{-1} \hat{u}_{q/p}^{-1},$$

$$\text{where } \hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \dots a^{\epsilon_{p-2}} b^{\epsilon_{p-1}}.$$

Remark 3.2. For $r = 0/1$ and $r = 1/0$, we have $u_{0/1} = ab$ and $u_{1/0} = 1$.

Now we define the sequences $S(r)$ and $T(r)$ and the cyclic sequences $CS(r)$ and $CT(r)$, all of which are read from u_r defined in Lemma 3.1, and review several important properties of these sequences from [6] so that we can adopt small cancellation theory. To this end we fix some definitions and notation. Let X be a set. By a *word* in X , we mean a finite sequence $x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}$ where $x_i \in X$ and $\epsilon_i = \pm 1$. Here we call $x_i^{\epsilon_i}$ the *i-th letter* of the word. For two words u, v in X , by $u \equiv v$ we denote the *visual equality* of u and v , meaning that if $u = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ and $v = y_1^{\delta_1} \dots y_m^{\delta_m}$ ($x_i, y_j \in X$; $\epsilon_i, \delta_j = \pm 1$), then $n = m$ and $x_i = y_i$ and $\epsilon_i = \delta_i$ for each $i = 1, \dots, n$. For example, two words $x_1 x_2 x_2^{-1} x_3$ and $x_1 x_3$ ($x_i \in X$) are *not* visually equal, though $x_1 x_2 x_2^{-1} x_3$ and $x_1 x_3$ are equal as elements of the free group with basis X . The length of a word v is denoted by $|v|$. A word v in X is said to be *reduced* if v does not contain xx^{-1} or $x^{-1}x$ for any $x \in X$. A word is said to be *cyclically reduced* if all its cyclic permutations are reduced. A *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (v) we denote the cyclic word associated with a cyclically reduced word v . Also by $(u) \equiv (v)$ we mean the *visual equality* of two cyclic words (u) and (v) . In fact, $(u) \equiv (v)$ if and only if v is visually a cyclic shift of u .

Definition 3.3. (1) Let v be a reduced word in $\{a, b\}$. Decompose v into

$$v \equiv v_1 v_2 \dots v_t,$$

where, for each $i = 1, \dots, t-1$, all letters in v_i have positive (resp., negative) exponents, and all letters in v_{i+1} have negative (resp., positive) exponents. Then the sequence of positive integers $S(v) := (|v_1|, |v_2|, \dots, |v_t|)$ is called the *S-sequence of v* .

(2) Let (v) be a cyclic word in $\{a, b\}$. Decompose (v) into

$$(v) \equiv (v_1 v_2 \cdots v_t),$$

where all letters in v_i have positive (resp., negative) exponents, and all letters in v_{i+1} have negative (resp., positive) exponents (taking subindices modulo t). Then the *cyclic* sequence of positive integers $CS(v) := (|v_1|, |v_2|, \dots, |v_t|)$ is called the *cyclic S -sequence of (v)* . Here the double parentheses denote that the sequence is considered modulo cyclic permutations.

(3) A reduced word v in $\{a, b\}$ is said to be *alternating* if $a^{\pm 1}$ and $b^{\pm 1}$ appear in v alternately, i.e., neither $a^{\pm 2}$ nor $b^{\pm 2}$ appears in v . A cyclic word (v) is said to be *alternating* if all cyclic permutations of v are alternating. In the latter case, we also say that v is *cyclically alternating*.

Definition 3.4. For a rational number r with $0 < r \leq 1$, let u_r be defined as in Lemma 3.1. Then the symbol $S(r)$ (resp., $CS(r)$) denotes the S -sequence $S(u_r)$ of u_r (resp., cyclic S -sequence $CS(u_r)$ of (u_r)), which is called the *S -sequence of slope r* (resp., the *cyclic S -sequence of slope r*).

Throughout this paper unless specified otherwise, we suppose that r is a rational number with $0 < r \leq 1$, and write r as a continued fraction:

$$r = [m_1, m_2, \dots, m_k] := \frac{1}{m_1 + \frac{1}{m_2 + \cdots + \frac{1}{m_k}}},$$

where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$ unless $k = 1$. For brevity, we write m for m_1 .

Lemma 3.5 ([6, Proposition 4.3]). *The following hold.*

- (1) Suppose $k = 1$, i.e., $r = 1/m$. Then $S(r) = (m, m)$.
- (2) Suppose $k \geq 2$. Then each term of $S(r)$ is either m or $m + 1$, and $S(r)$ begins with $m + 1$ and ends with m . Moreover, the following hold.
 - (a) If $m_2 = 1$, then no two consecutive terms of $S(r)$ can be (m, m) , so there is a sequence of positive integers (t_1, t_2, \dots, t_s) such that

$$S(r) = (t_1 \langle m + 1 \rangle, m, t_2 \langle m + 1 \rangle, m, \dots, t_s \langle m + 1 \rangle, m).$$

Here, the symbol " $t_i \langle m + 1 \rangle$ " represents t_i successive $m + 1$'s.

- (b) If $m_2 \geq 2$, then no two consecutive terms of $S(r)$ can be $(m + 1, m + 1)$, so there is a sequence of positive integers (t_1, t_2, \dots, t_s) such that

$$S(r) = (m + 1, t_1 \langle m \rangle, m + 1, t_2 \langle m \rangle, \dots, m + 1, t_s \langle m \rangle).$$

Here, the symbol “ $t_i\langle m \rangle$ ” represents t_i successive m 's.

Definition 3.6. If $k \geq 2$, the symbol $T(r)$ denotes the sequence (t_1, t_2, \dots, t_s) in Lemma 3.5, which is called the *T-sequence of slope r* . The symbol $CT(r)$ denotes the cyclic sequence represented by $T(r)$, which is called the *cyclic T-sequence of slope r* .

Example 3.7. (1) Let $r = 10/37 = [3, 1, 2, 3]$. By Lemma 3.1, we see that the S -sequence of \hat{u}_r is

$$S(\hat{u}_r) = (3, 4, 4, 3, 4, 4, 3, 4, 4, 3).$$

By the formula for u_r in Lemma 3.1, this implies

$$S(r) = S(u_r) = (\underbrace{4, 4, 4}_3, 3, \underbrace{4, 4}_2, 3, \underbrace{4, 4}_2, 3, \underbrace{4, 4, 4}_3, 3, \underbrace{4, 4}_2, 3, \underbrace{4, 4}_2, 3).$$

So $T(r) = (3, 2, 2, 3, 2, 2)$ and $CT(r) = ((3, 2, 2, 3, 2, 2))$.

(2) Let $r = 8/35 = [4, 2, 1, 2]$. Again by Lemma 3.1, we obtain that the S -sequence of \hat{u}_r is

$$S(\hat{u}_r) = (4, 4, 5, 4, 4, 5, 4, 4).$$

By the formula for u_r in Lemma 3.1, this implies

$$S(r) = S(u_r) = (5, \underbrace{4}_1, 5, \underbrace{4, 4}_2, 5, \underbrace{4, 4}_2, 5, \underbrace{4}_1, 5, \underbrace{4, 4}_2, 5, \underbrace{4, 4}_2).$$

So $T(r) = (1, 2, 2, 1, 2, 2)$ and $CT(r) = ((1, 2, 2, 1, 2, 2))$.

Lemma 3.8 ([6, Proposition 4.4 and Corollary 4.6]). *Let \tilde{r} be the rational number defined as*

$$\tilde{r} = \begin{cases} [m_3, \dots, m_k] & \text{if } m_2 = 1; \\ [m_2 - 1, m_3, \dots, m_k] & \text{if } m_2 \geq 2. \end{cases}$$

Then we have $CS(\tilde{r}) = CT(r)$.

Lemma 3.9 ([6, Proposition 4.5]). *The sequence $S(r)$ has a decomposition (S_1, S_2, S_1, S_2) which satisfies the following.*

- (1) *Each S_i is symmetric, i.e., the sequence obtained from S_i by reversing the order is equal to S_i . (Here, S_1 is empty if $k = 1$.)*
- (2) *Each S_i (if it is not empty) occurs only twice in the cyclic sequence $CS(r)$.*
- (3) *S_1 (if it is not empty) begins and ends with $m + 1$.*
- (4) *S_2 begins and ends with m .*

Example 3.10. (1) Let $r = 10/37 = [3, 1, 2, 3]$. Recall from Example 3.7 that

$$S(r) = (4, 4, 4, 3, 4, 4, 3, 4, 4, 3, 4, 4, 4, 3, 4, 4, 3, 4, 4, 3).$$

Putting $S_1 = (4, 4, 4)$ and $S_2 = (3, 4, 4, 3, 4, 4, 3)$, we have

$$S(r) = (S_1, S_2, S_1, S_2),$$

where S_1 and S_2 satisfy all the assertions in Lemma 3.9.

(2) Let $r = 8/35 = [4, 2, 1, 2]$. Recall also from Example 3.7 that

$$S(r) = (5, 4, 5, 4, 4, 5, 4, 4, 5, 4, 5, 4, 4, 5, 4, 4).$$

Putting $S_1 = (5, 4, 5)$ and $S_2 = (4, 4, 5, 4, 4)$, we also have

$$S(r) = (S_1, S_2, S_1, S_2),$$

where S_1 and S_2 satisfy all the assertions in Lemma 3.9.

Remark 3.11. By using the fact that u_r is obtained from the line of slope r in $\mathbb{R}^2 - \mathbb{Z}^2$ by reading its intersection with the vertical lattice lines, we see that the slope $s = q/p$ is recovered from $CS(s) = ((S_1, S_2, S_1, S_2))$ by the rule that p is the sum of the terms of S_1 and S_2 whereas q is the sum of the lengths of S_1 and S_2 .

Lemma 3.12 ([6, Proof of Proposition 4.5]). *Let \tilde{r} be the rational number defined as in Lemma 3.8. Also let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ and $S(r) = (S_1, S_2, S_1, S_2)$ be decompositions described as in Lemma 3.9. Then the following hold.*

- (1) *If $m_2 = 1$ and $k = 3$, then $T_1 = \emptyset$, $T_2 = (m_3)$, and $S_1 = (m_3\langle m+1 \rangle)$, $S_2 = (m)$.*
- (2) *If $m_2 = 1$ and $k \geq 4$, then $T_1 = (t_1, \dots, t_{s_1})$, $T_2 = (t_{s_1+1}, \dots, t_{s_2})$, and $S_1 = (t_1\langle m+1 \rangle, m, t_2\langle m+1 \rangle, \dots, t_{s_1-1}\langle m+1 \rangle, m, t_{s_1}\langle m+1 \rangle)$, $S_2 = (m, t_{s_1+1}\langle m+1 \rangle, m, \dots, m, t_{s_2}\langle m+1 \rangle, m)$.*
- (3) *If $k = 2$, then $T_1 = \emptyset$, $T_2 = (m_2 - 1)$, and $S_1 = (m+1)$, $S_2 = ((m_2 - 1)\langle m \rangle)$.*
- (4) *If $m_2 \geq 2$ and $k \geq 3$, then $T_1 = (t_1, \dots, t_{s_1})$, $T_2 = (t_{s_1+1}, \dots, t_{s_2})$, and $S_1 = (m+1, t_{s_1+1}\langle m \rangle, m+1, \dots, m+1, t_{s_2}\langle m \rangle, m+1)$, $S_2 = (t_1\langle m \rangle, m+1, t_2\langle m \rangle, \dots, t_{s_1-1}\langle m \rangle, m+1, t_{s_1}\langle m \rangle)$.*

Example 3.13. (1) If $r = [2, 1, 5]$, then $\tilde{r} = [5]$ by Lemma 3.8. So by Lemma 3.5(1), $S(\tilde{r}) = (\emptyset, 5, \emptyset, 5)$. Thus by Lemma 3.12(1),

$$S(r) = (5\langle 3 \rangle, 2, 5\langle 3 \rangle, 2),$$

where $S_1 = (5\langle 3 \rangle)$ and $S_2 = (2)$.

(2) If $r = [2, 5]$, then $\tilde{r} = [4]$ by Lemma 3.8. So by Lemma 3.5(1), $S(\tilde{r}) = (\emptyset, 4, \emptyset, 4)$. Thus by Lemma 3.12(3),

$$S(r) = (3, 4\langle 2 \rangle, 3, 4\langle 2 \rangle),$$

where $S_1 = (3)$ and $S_2 = (4\langle 2 \rangle)$.

By Lemmas 3.5 and 3.12, we easily obtain the following corollary.

Corollary 3.14. *Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Lemma 3.9. Then the following hold.*

- (1) *If $m_2 = 1$, then $(m+1, m+1)$ appears in S_1 .*
- (2) *If $m_2 \geq 2$ and if $r \neq [m, 2] = 2/(2m+1)$, then (m, m) appears in S_2 .*

4. SMALL CANCELLATION THEORY

4.1. Basic definitions and preliminary facts

Let $F(X)$ be the free group with basis X . A subset R of $F(X)$ is said to be *symmetrized*, if all elements of R are cyclically reduced and, for each $w \in R$, all cyclic permutations of w and w^{-1} also belong to R .

Definition 4.1. Suppose that R is a symmetrized subset of $F(X)$. A nonempty word b is called a *piece* if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv bc_1$ and $w_2 \equiv bc_2$. The small cancellation conditions $C(p)$ and $T(q)$, where p and q are integers such that $p \geq 2$ and $q \geq 3$, are defined as follows (see [15]).

- (1) Condition $C(p)$: If $w \in R$ is a product of t pieces, then $t \geq p$.
- (2) Condition $T(q)$: For $w_1, \dots, w_t \in R$ with no successive elements w_i, w_{i+1} an inverse pair ($i \bmod t$), if $t < q$, then at least one of the products $w_1w_2, \dots, w_{t-1}w_t, w_tw_1$ is freely reduced without cancellation.

We recall the following lemma which concerns the word u_r defined in Lemma 3.1.

Lemma 4.2 ([6, Lemma 5.3]). *Suppose that $r = [m_1, \dots, m_k]$ is a rational number with $0 < r < 1$, and let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Lemma 3.9. Decompose*

$$u_r \equiv v_1v_2v_3v_4,$$

where $S(v_1) = S(v_3) = S_1$ and $S(v_2) = S(v_4) = S_2$. Then the following hold.

- (1) *If $k = 1$, then the following hold.*
 - (a) *No piece can contain v_2 or v_4 .*
 - (b) *No piece is of the form $v_{2e}v_{4b}$ or $v_{4e}v_{2b}$, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i , respectively.*

- (c) Every subword of the form v_{2b} , v_{2e} , v_{4b} , or v_{4e} is a piece, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i with $|v_{ib}|, |v_{ie}| \leq |v_i| - 1$, respectively.
- (2) If $k \geq 2$, then the following hold.
 - (a) No piece can contain v_1 or v_3 .
 - (b) No piece is of the form $v_{1e}v_2v_{3b}$ or $v_{3e}v_4v_{1b}$, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i , respectively.
 - (c) Every subword of the form $v_{1e}v_2$, v_2v_{3b} , $v_{3e}v_4$, or v_4v_{1b} is a piece, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i with $|v_{ib}|, |v_{ie}| \leq |v_i| - 1$, respectively.

By using the above lemma, we established in [10] the following key lemma concerning the cyclic word (u_r^n) .

Lemma 4.3 ([10, Lemma 4.3]). *Suppose that r is a rational number with $0 < r < 1$ and that $n \geq 2$ is an integer. Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Lemma 3.9. Then the following hold.*

- (1) The cyclic word (u_r^n) is not a product of t pieces with $t \leq 4n - 1$.
- (2) Let w be a subword of the cyclic word (u_r^n) which is a product of $4n - 1$ pieces but is not a product of t pieces with $t < 4n - 1$. Then w contains a subword, w' , such that $S(w') = ((2n - 1)\langle S_1, S_2 \rangle, \ell)$ or $S(w') = (\ell, (2n - 1)\langle S_2, S_1 \rangle)$, where $\ell \in \mathbb{Z}_+$.

The following proposition enables us to apply the small cancellation theory to our problem.

Proposition 4.4 ([10, Proposition 4.4]). *Suppose that r is a rational number with $0 < r < 1$ and that n is an integer with $n \geq 2$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r^n of the upper presentation of $G(r; n)$. Then R satisfies $C(4n)$ and $T(4)$.*

Now we want to investigate the geometric consequences of Proposition 4.4. Let us begin with necessary definitions and notation following [15]. A map M is a finite 2-dimensional cell complex embedded in \mathbb{R}^2 . To be precise, M is a finite collection of vertices (0-cells), edges (1-cells), and faces (2-cells) in \mathbb{R}^2 satisfying the following conditions.

- (i) A vertex is a point in \mathbb{R}^2 .
- (ii) An edge e is homeomorphic to an open interval such that $\bar{e} = e \cup \{a\} \cup \{b\}$, where a and b are vertices of M which are possibly identical.
- (iii) For each face D of M , there is a continuous map f from the 2-ball B^2 to \mathbb{R}^2 such that

- (a) the restriction of f to the interior of B^2 is a homeomorphism onto D , and
- (b) the image of ∂B^2 is equal to $\cup_{i=1}^t \bar{e}_i$ for some set $\{e_1, \dots, e_t\}$ of edges of M .

The underlying space of M , i.e., the union of the cells in M , is also denoted by the same symbol M . The boundary (frontier), ∂M , of M in \mathbb{R}^2 is regarded as a 1-dimensional subcomplex of M . An edge may be traversed in either of two directions. If v is a vertex of a map M , $d_M(v)$, the *degree of v* , denotes the number of oriented edges in M having v as initial vertex. A vertex v of M is called an *interior vertex* if $v \notin \partial M$, and an edge e of M is called an *interior edge* if $e \not\subset \partial M$.

A *path* in M is a sequence of oriented edges e_1, \dots, e_t such that the initial vertex of e_{i+1} is the terminal vertex of e_i for every $1 \leq i \leq t-1$. A *cycle* is a closed path, namely a path e_1, \dots, e_t such that the initial vertex of e_1 is the terminal vertex of e_t . If D is a face of M , any cycle of minimal length which includes all the edges of the boundary, ∂D , of D going around once along the boundary of D is called a *boundary cycle* of D . To be precise it is defined as follows. Let $f : B^2 \rightarrow D$ be a continuous map satisfying the condition (iii) above. We may assume that ∂B^2 has a cellular structure such that the restriction of f to each cell is a homeomorphism. Choose an arbitrary orientation of ∂B^2 , and let $\hat{e}_1, \dots, \hat{e}_t$ be the oriented edges of ∂B^2 , which are oriented in accordance with the orientation of ∂B^2 and which lie on ∂B^2 in this cyclic order with respect to the orientation of ∂B^2 . Let e_i be the orientated edge $f(\hat{e}_i)$ of M . Then the cycle e_1, \dots, e_t is a boundary cycle of D . By $d_M(D)$, the *degree of D* , we denote the number of unoriented edges in a boundary cycle of D .

Definition 4.5. A non-empty map M is called a $[p, q]$ -map if the following conditions hold.

- (i) $d_M(v) \geq p$ for every interior vertex v in M .
- (ii) $d_M(D) \geq q$ for every face D in M .

Definition 4.6. Let R be a symmetrized subset of $F(X)$. An R -diagram is a map M and a function ϕ assigning to each oriented edge e of M , as a *label*, a reduced word $\phi(e)$ in X such that the following hold.

- (i) If e is an oriented edge of M and e^{-1} is the oppositely oriented edge, then $\phi(e^{-1}) = \phi(e)^{-1}$.

- (ii) For any boundary cycle δ of any face of M , $\phi(\delta)$ is a cyclically reduced word representing an element of R . (If $\alpha = e_1, \dots, e_t$ is a path in M , we define $\phi(\alpha) \equiv \phi(e_1) \cdots \phi(e_t)$.)

Let D_1 and D_2 be faces (not necessarily distinct) of M with an edge $e \subseteq \partial D_1 \cap \partial D_2$. Let $e\delta_1$ and $\delta_2 e^{-1}$ be boundary cycles of D_1 and D_2 , respectively. Let $\phi(\delta_1) = f_1$ and $\phi(\delta_2) = f_2$. An R -diagram M is said to be *reduced* if one never has $f_2 = f_1^{-1}$. It should be noted that if M is reduced then $\phi(e)$ is a piece for every interior edge e of M .

As explained in [6, Convention 1], we may assume the following convention.

Convention 4.7. Suppose that r is a rational number with $0 < r < 1$ and that n is an integer with $n \geq 2$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r^n of the upper presentation of $G(r; n)$. For any reduced R -diagram M , we assume that M satisfies the following.

- (1) $d_M(v) \geq 3$ for every interior vertex v of M .
- (2) For every edge e of ∂M , the label $\phi(e)$ is a piece.
- (3) For a path e_1, \dots, e_t in ∂M of length $t \geq 2$ such that the vertex $\bar{e}_i \cap \bar{e}_{i+1}$ has degree 2 for $i = 1, 2, \dots, t-1$, $\phi(e_1)\phi(e_2) \cdots \phi(e_t)$ cannot be expressed as a product of fewer than t pieces.

The following corollary is immediate from Proposition 4.4 and Convention 4.7.

Corollary 4.8 ([10, Corollary 4.9]). *Suppose that r is a rational number with $0 < r < 1$, and that n is an integer with $n \geq 2$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r^n of the upper presentation of $G(r; n)$. Then every reduced R -diagram is a $[4, 4n]$ -map.*

We turn to interpreting conjugacy in terms of diagrams.

Definition 4.9. An *annular map* M is a connected map such that $\mathbb{R}^2 - M$ has exactly two connected components. It is said to be *nontrivial* if it contains a 2-cell. For a symmetrized subset R of $F(a, b)$, an *annular R -diagram* is an R -diagram whose underlying map is an annular map.

Let M be an annular R -diagram, and let K and H be, respectively, the unbounded and bounded components of $\mathbb{R}^2 - M$. We call $\partial K (\subset \partial M)$ the *outer boundary* of M , while $\partial H (\subset \partial M)$ is called the *inner boundary* of M . Clearly, the *boundary* of M , ∂M , is the union of the outer boundary and the inner boundary. A cycle of minimal length which contains all the edges in the outer (inner, resp.) boundary of M going around once along the boundary of

K (H , resp.) is an *outer* (*inner*, resp.) *boundary cycle* of M . An *outer* (*inner*, resp.) *boundary label* of M is defined to be a word $\phi(\alpha)$ in X for α an outer (inner, resp.) boundary cycle of M .

Convention 4.10. Since M is embedded in \mathbb{R}^2 , each 2-cell of M inherits an orientation of \mathbb{R}^2 . Throughout this paper, we assume, unlike the usual orientation convention, that \mathbb{R}^2 is oriented so that the boundary cycles of the 2-cells of M are clockwise. Thus the outer boundary cycles are clockwise and inner boundary cycles are counterclockwise, unlike the convention in [15, p.253].

The following lemma is a well-known classical result in combinatorial group theory.

Lemma 4.11 ([15, Lemmas V.5.1 and V.5.2]). *Suppose $G = \langle X \mid R \rangle$ with R being symmetrized. Let u, v be two cyclically reduced words in X which are not trivial in G and which are not conjugate in $F(X)$. Then u and v represent conjugate elements in G if and only if there exists a reduced nontrivial annular R -diagram M such that u is an outer boundary label and v^{-1} is an inner boundary label of M .*

4.2. Structure theorem and its corollary

We recall the following lemma obtained from the arguments of [15, Theorem V.3.1].

Lemma 4.12. *Let M be an arbitrary connected map, and let h denote the number of holes of M . Then*

$$4 - 4h \leq \sum_{v \in \partial M} (3 - d_M(v)) + \sum_{v \in M - \partial M} (4 - d_M(v)) + \sum_{D \in M} (4 - d_M(D)).$$

In particular, if M is a $[4, 4n]$ -map, then

$$4 - 4h \leq \sum_{v \in \partial M} (3 - d_M(v)) + \sum_{D \in M} (4 - 4n).$$

In the above lemma and throughout this paper, the symbol $v \in X$ ($D \in X$, resp.) under the symbol \sum , where X is a map M or a subspace of a map M , means that the sum is over the vertices v (the faces D , resp.) of the map M contained in the subspace X .

The following proposition will play an essential role in the proof of the structure theorem.

Proposition 4.13. *Let M be an arbitrary connected $[4, 4n]$ -map. Put*

- h = the number of holes of M ;
- V = the number of vertices of M ;
- E = the number of (unoriented) edges of M ;
- F = the number of faces of M .

Also put

- A = the number of vertices v in ∂M such that $d_M(v) = 2$;
- B = the number of vertices v in ∂M such that $d_M(v) \geq 4$;
- C = the number of vertices v in ∂M such that $d_M(v) = 3$.

Then the following hold, where $\lceil x \rceil$ is the smallest integer not less than x .

- (1) $F \geq B + \lceil C/2 \rceil + 1 - h$.
- (2) $A \geq (4n - 3)B + (4n - 4)\lceil C/2 \rceil + 4(1 - h)n$.

Proof. (1) Since M is a $[4, 4n]$ -map, every interior vertex of M has degree at least 4. So we have

$$E \geq 1/2\{2A + 3C + 4(V - A - C)\} = 2V - A - C/2.$$

This inequality together with Euler's formula $V - E + F = 1 - h$ yields $V + F \geq 2V - A - C/2 + 1 - h$, so that

$$\begin{aligned} F &\geq V - A - C/2 + 1 - h \\ &\geq (A + B + C) - A - C/2 + 1 - h \\ &= B + C/2 + 1 - h. \end{aligned}$$

Since F is an integer, we finally have

$$F \geq B + \lceil C/2 \rceil + 1 - h,$$

as required.

(2) By Lemma 4.12, we have

$$\begin{aligned} 4 - 4h &\leq \sum_{v \in \partial M} (3 - d_M(v)) + \sum_{D \in M} (4 - 4n) \\ &= \sum_{v \in \partial M} (3 - d_M(v)) + F(4 - 4n), \end{aligned}$$

so that

$$\sum_{v \in \partial M} (3 - d_M(v)) \geq F(4n - 4) + 4 - 4h.$$

Here, since $A - B \geq \sum_{v \in \partial M} (3 - d_M(v))$, we have, by (1),

$$\begin{aligned} A - B &\geq F(4n - 4) + 4 - 4h \\ &\geq (B + \lceil C/2 \rceil + 1 - h)(4n - 4) + 4 - 4h \\ &= (4n - 4)B + (4n - 4)\lceil C/2 \rceil + 4(1 - h)n, \end{aligned}$$

so that $A \geq (4n - 3)B + (4n - 4)\lceil C/2 \rceil + 4(1 - h)n$, as required. \square

Remark 4.14. In Proposition 4.13(2), if the equality holds, then the following hold.

- (1) $V = A + B + C$, that is, there is no vertex of M of degree 1 and every vertex of M lies in ∂M ;
- (2) $\sum_{v \in \partial M} (3 - d_M(v)) = A - B$, that is, every vertex of ∂M has degree 2, 3 or 4;
- (3) $d_M(D) = 4n$ for every face D of M .

Now we obtain the following strong structure theorem.

Theorem 4.15 (Structure Theorem). *Suppose that $r = [m_1, \dots, m_k]$ is a rational number with $0 < r < 1$, and that n is an integer with $n \geq 2$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r^n of the upper presentation of $G(r; n)$, and let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Lemma 3.9. Suppose that M is a reduced nontrivial annular R -diagram such that, for α and δ which are, respectively, arbitrary outer and inner boundary cycles of M ,*

- (i) *the words $\phi(\alpha)$ and $\phi(\delta)$ are cyclically reduced;*
- (ii) *the words $\phi(\alpha)$ and $\phi(\delta)$ are cyclically alternating;*
- (iii) *if $k = 1$, then $CS(\phi(\alpha))$ and $CS(\phi(\delta))$ do not contain $((2n - 2)\langle m \rangle)$ as a subsequence;*
- (iv) *if $k \geq 2$, then $CS(\phi(\alpha))$ and $CS(\phi(\delta))$ do not contain $((2n - 1)\langle S_1, S_2 \rangle)$ nor $((2n - 1)\langle S_2, S_1 \rangle)$ as a subsequence.*

Let the outer and inner boundaries of M be denoted by σ and τ , respectively. Then the following hold.

- (1) *The outer and inner boundaries σ and τ are simple, i.e., they are homeomorphic to the circle, and there is no edge contained in $\sigma \cap \tau$.*
- (2) *Every vertex of M lies in ∂M .*
- (3) *$d_M(v) = 2$ or 4 for every vertex v of ∂M .*
- (4) *In particular, if $\sigma \cap \tau = \emptyset$, then between any two vertices of degree 4 there should occur exactly $4n - 3$ vertices of degree 2 on both σ and τ , and $d_M(D) = 4n$ for every face D of M .*

In the above theorem, we mean by a *subsequence* a subsequence without leap. Namely a sequence (a_1, a_2, \dots, a_p) is called a *subsequence* of a cyclic sequence, if there is a sequence (b_1, b_2, \dots, b_t) representing the cyclic sequence such that $p \leq t$ and $a_i = b_i$ for $1 \leq i \leq p$.

Before proving Theorem 4.15, we prepare the following lemma.

Lemma 4.16. *Under the assumption of Theorem 4.15, the outer and inner boundaries σ and τ are simple.*

Proof. Suppose on the contrary that σ or τ is not simple. Then there is an extremal disk, say J , which is properly contained in M and connected to the rest of M by a single vertex, say v_0 . Here, recall that an *extremal disk* of a map M is a submap of M which is topologically a disk and which has a boundary cycle e_1, \dots, e_t such that the edges e_1, \dots, e_t occur in order in some boundary cycle of the whole map M . By Corollary 4.8, M is a connected annular $[4, 4n]$ -map. Then J is a connected and simply-connected $[4, 4n]$ -map.

By assumptions (i) and (ii), there is no vertex of degree 1 nor 3 in ∂M . So every vertex in $\partial J - \{v_0\} \subseteq \partial M$ has degree 2 or at least 4. Put

A = the number of vertices v in $\partial J - \{v_0\}$ such that $d_J(v) = 2$;

B = the number of vertices v in $\partial J - \{v_0\}$ such that $d_J(v) \geq 4$.

By assumptions (iii) and (iv) together with Lemma 4.3, the word $\phi(\partial J|_{v_0})$ does not contain any subword of the cyclic word (u_r^n) which is a product of $4n - 1$ pieces but is not a product of less than $4n - 1$ pieces, where $\partial J|_{v_0}$ denotes a boundary cycle of J starting from the vertex v_0 . This implies by Convention 4.7(3) that there are no $4n - 2$ consecutive degree 2 vertices in $\partial J - \{v_0\}$. Hence we have $A \leq (4n - 3)(B + 1)$. We will derive a contradiction to this inequality using Proposition 4.13(2) applied to J .

Clearly $d_J(v_0) \geq 2$. First if $d_J(v_0) = 2$, then

$$A + 1 \geq (4n - 3)B + 4n = (4n - 3)(B + 1) + 3,$$

so that $A \geq (4n - 3)(B + 1) + 2$, contrary to $A \leq (4n - 3)(B + 1)$. Next if $d_J(v_0) = 3$, then

$$\begin{aligned} A &\geq (4n - 3)B + (4n - 4)\lceil 1/2 \rceil + 4n = (4n - 3)B + (4n - 4) + 4n \\ &= (4n - 3)(B + 1) + 4n - 1, \end{aligned}$$

contrary to $A \leq (4n - 3)(B + 1)$. Finally if $d_J(v_0) \geq 4$, then

$$A \geq (4n - 3)(B + 1) + 4n,$$

contrary to $A \leq (4n - 3)(B + 1)$. □

Proof of Theorem 4.15. Arguing as in the proof of Lemma 4.16, M is a connected annular $[4, 4n]$ -map such that every vertex in ∂M has degree 2 or at least 4. We divide into two cases.

Case 1. $\sigma \cap \tau = \emptyset$.

In this case, (1) follows immediately from Lemma 4.16.

(2)–(4) Put

- A_1 = the number of vertices v in σ such that $d_M(v) = 2$;
- A_2 = the number of vertices v in τ such that $d_M(v) = 2$;
- B_1 = the number of vertices v in σ such that $d_M(v) \geq 4$;
- B_2 = the number of vertices v in τ such that $d_M(v) \geq 4$.

Since ∂M is the disjoint union of σ and τ , Proposition 4.13(2) applied to M yields

$$A_1 + A_2 \geq (4n - 3)(B_1 + B_2) = (4n - 3)B_1 + (4n - 3)B_2.$$

Furthermore, since $CS(\phi(\alpha))$ and $CS(\phi(\delta))$ do not contain $((2n - 1)\langle S_1, S_2 \rangle)$ nor $((2n - 1)\langle S_2, S_1 \rangle)$ as a subsequence, Lemma 4.3 together with Convention 4.7(3) implies that there are no $4n - 2$ consecutive degree 2 vertices on σ nor on τ , so that we have $A_1 \leq (4n - 3)B_1$ and $A_2 \leq (4n - 3)B_2$. Hence the above inequality is actually an equality. By Remark 4.14, every vertex of M lies in ∂M , every vertex in ∂M has degree 2 or 4, and $d_M(D) = 4n$ for every face D of M . Moreover, since $A_1 = (4n - 3)B_1$ and $A_2 = (4n - 3)B_2$ and since there are no $4n - 2$ consecutive degree 2 vertices on both σ and τ , between any two vertices of degree 4 there should occur exactly $4n - 3$ vertices of degree 2 on both σ and τ . Therefore (2)–(4) hold.

Case 2. $\sigma \cap \tau \neq \emptyset$.

(1) Suppose on the contrary that $\sigma \cap \tau$ contains an edge. As illustrated in Figure 2, there is a submap J of M such that

- (i) J is bounded by a simple closed path of the form $\sigma_1\tau_1$, where $\sigma_1 \subseteq \sigma$ and $\tau_1 \subseteq \tau$;
- (ii) J is connected to the rest of M by two distinct vertices, say v_1 and v_2 , where $\sigma_1 \cap \tau_1 = \{v_1, v_2\}$ and v_1 is an endpoint of an edge contained in $\sigma \cap \tau$. Note that $d_J(v_1) = d_M(v_1) - 1 \geq 3$ and $d_J(v_2) \geq 2$.

Then J is a connected and simply connected $[4, 4n]$ -map such that every vertex in $\partial J - \{v_1, v_2\}$ has degree 2 or at least 4. Put

A = the number of vertices v in $\partial J - \{v_1, v_2\}$ such that $d_J(v) = 2$;

B = the number of vertices v in $\partial J - \{v_1, v_2\}$ such that $d_J(v) \geq 4$.

Arguing as in Case 1, there are no $4n - 2$ consecutive degree 2 vertices on $\sigma_1 - \{v_1, v_2\}$ nor on $\tau_1 - \{v_1, v_2\}$. Hence we have $A \leq (4n - 3)(B + 2)$. We will derive a contradiction to this inequality using Proposition 4.13(2) applied to J .

First if $d_J(v_1) = 3$ and $d_J(v_2) = 2$, then

$$\begin{aligned} A + 1 &\geq (4n - 3)B + (4n - 4)\lceil 1/2 \rceil + 4n = (4n - 3)B + (4n - 4) + 4n \\ &= (4n - 3)(B + 2) + 2, \end{aligned}$$

so that $A \geq (4n - 3)(B + 2) + 1$, contrary to $A \leq (4n - 3)(B + 2)$. Second if $d_J(v_1) \geq 4$ and $d_J(v_2) = 2$, then

$$A + 1 \geq (4n - 3)(B + 1) + 4n = (4n - 3)(B + 2) + 3,$$

so that $A \geq (4n - 3)(B + 2) + 2$, contrary to $A \leq (4n - 3)(B + 2)$. Third if $d_J(v_1) = 3$ and $d_J(v_2) = 3$, then

$$\begin{aligned} A &\geq (4n - 3)B + (4n - 4)\lceil 1 \rceil + 4n = (4n - 3)B + (4n - 4) + 4n \\ &= (4n - 3)(B + 2) + 2, \end{aligned}$$

contrary to $A \leq (4n - 3)(B + 2)$. Fourth either if $d_J(v_1) \geq 4$ and $d_J(v_2) = 3$ or if $d_J(v_1) = 3$ and $d_J(v_2) \geq 4$, then

$$\begin{aligned} A &\geq (4n - 3)(B + 1) + (4n - 4)\lceil 1/2 \rceil + 4n = (4n - 3)(B + 1) + (4n - 4) + 4n \\ &= (4n - 3)(B + 2) + 4n - 1, \end{aligned}$$

contrary to $A \leq (4n - 3)(B + 2)$. Finally if $d_J(v_1) \geq 4$ and $d_J(v_2) \geq 4$, then

$$A \geq (4n - 3)(B + 2) + 4n,$$

contrary to $A \leq (4n - 3)(B + 2)$.

(2)–(3) By (1), $\sigma \cap \tau$ consists of finitely many vertices in M . First suppose that $\sigma \cap \tau$ consists of at least two vertices, say v_1, \dots, v_t , where $t \geq 2$ and where these vertices are indexed so that there is a submap J_i of M for every $i = 1, \dots, t$ such that

- (i) J_i is bounded by a simple closed path of the form $\sigma_i \tau_i$, where $\sigma_i \subseteq \sigma$ and $\tau_i \subseteq \tau$;

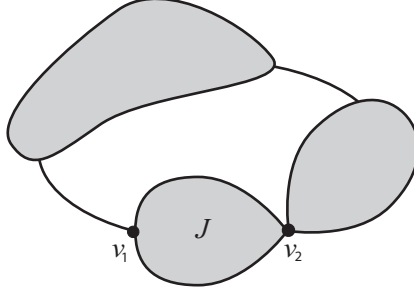


FIGURE 2. A possible annular map M when $\sigma \cap \tau$ contains an edge.

- (ii) J_i is connected to the rest of M by two distinct vertices, say v_i and v_{i+1} , where $\sigma_i \cap \tau_i = \{v_i, v_{i+1}\}$ and where $d_{J_i}(v_i), d_{J_i}(v_{i+1}) \geq 2$ and $d_{J_i}(v_{i+1}) + d_{J_{i+1}}(v_{i+1}) = d_M(v_{i+1})$ (taking the indices modulo n).

Then each J_i is a connected and simply connected $[4, 4n]$ -map such that $M = J_1 \cup \dots \cup J_t$. Moreover $\sigma = \sigma_1 \cup \dots \cup \sigma_t$ and $\tau = \tau_1 \cup \dots \cup \tau_t$. The same argument as for (M', v'_0, v''_0) below applies to each (J_i, v_i, v_{i+1}) to prove the assertions.

Next suppose that $\sigma \cap \tau$ consists of a single vertex, say v_0 . Cut M open at v_0 to get a connected and simply connected $[4, 4n]$ -map M' . In this process, the vertex v_0 is separated into two distinct vertices, say v'_0 and v''_0 , in M' such that $d_{M'}(v'_0), d_{M'}(v''_0) \geq 2$ and $d_{M'}(v'_0) + d_{M'}(v''_0) = d_M(v_0)$. Then M' is bounded by a simple closed path of the form $\sigma_0 \tau_0$, where $\sigma_0 \cap \tau_0 = \{v'_0, v''_0\}$. Put

F = the number of faces of M' ;

A = the number of vertices v in $\partial M' - \{v'_0, v''_0\}$ such that $d_{M'}(v) = 2$;

B = the number of vertices v in $\partial M' - \{v'_0, v''_0\}$ such that $d_{M'}(v) \geq 4$.

Claim 1. $d_{M'}(v'_0) = d_{M'}(v''_0) = 2$.

Proof of Claim 1. Since every vertex in $\partial M' - \{v'_0, v''_0\}$ has degree 2 or at least 4 and since $4n - 2$ vertices of degree 2 do not occur consecutively on $\sigma_0 - \{v'_0, v''_0\}$ nor on $\tau_0 - \{v'_0, v''_0\}$, we have $A \leq (4n - 3)(B + 2)$. Suppose on the contrary that $d_{M'}(v'_0) \geq 3$ or $d_{M'}(v''_0) \geq 3$. Without loss of generality, assume that $d_{M'}(v'_0) \geq 3$. Then repeating the same arguments as in the proof of (1) replacing J , v_1 and v_2 with M' , v'_0 and v''_0 , respectively, we obtain a contradiction to $A \leq (4n - 3)(B + 2)$. \square

By Claim 1, $d_{M'}(v'_0) = d_{M'}(v''_0) = 2$. Hence there exist unique 2-cells D_1 and D_2 in M' such that $v'_0 \in \partial D_1$ and $v''_0 \in \partial D_2$.

Claim 2. $D_1 = D_2$.

Proof of Claim 2. Suppose on the contrary that $D_1 \neq D_2$. Then note that the number of vertices of degree 2 on $(\partial D_1 \cup \partial D_2) \cap (\sigma_0 \cup \tau_0) - \{v'_0, v''_0\}$ is less than or equal to $d_{M'}(D_1) + d_{M'}(D_2) - 6$. Since every vertex in $\partial M' - \{v'_0, v''_0\}$ has degree 2 or at least 4 and since $4n - 2$ vertices of degree 2 do not occur consecutively on $\sigma_0 - \{v'_0, v''_0\}$ nor on $\tau_0 - \{v'_0, v''_0\}$, we have

$$A \leq (4n - 3)(B - 2) + d_{M'}(D_1) + d_{M'}(D_2) - 6.$$

On the other hand, by Lemma 4.12,

$$\begin{aligned} 4 &\leq \sum_{v \in \partial M'} (3 - d_{M'}(v)) + \sum_{D \in M'} (4 - d_{M'}(D)) \\ &\leq \sum_{v \in \partial M'} (3 - d_{M'}(v)) + \sum_{D \in M' - \{D_1, D_2\}} (4 - 4n) + (4 - d_{M'}(D_1)) + (4 - d_{M'}(D_2)), \end{aligned}$$

so that

$$4 + \sum_{D \in M' - \{D_1, D_2\}} (4n - 4) \leq \sum_{v \in \partial M'} (3 - d_{M'}(v)) + (4 - d_{M'}(D_1)) + (4 - d_{M'}(D_2)).$$

By Proposition 4.13(1),

$$(B - 1)(4n - 4) \leq (F - 2)(4n - 4) = \sum_{D \in M' - \{D_1, D_2\}} (4n - 4);$$

hence

$$4 + (B - 1)(4n - 4) \leq \sum_{v \in \partial M'} (3 - d_{M'}(v)) + (4 - d_{M'}(D_1)) + (4 - d_{M'}(D_2)).$$

Note that $d_{M'}(v'_0) = d_{M'}(v''_0) = 2$ and so $\sum_{v \in \partial M'} (3 - d_{M'}(v)) \leq A + 2 - B$. Hence the above inequality implies

$$4 + (B - 1)(4n - 4) \leq (A + 2 - B) + (4 - d_{M'}(D_1)) + (4 - d_{M'}(D_2)).$$

Thus

$$\begin{aligned} A &\geq (B - 1)(4n - 4) + B - 6 + d_{M'}(D_1) + d_{M'}(D_2) \\ &= (4n - 3)(B - 2) + d_{M'}(D_1) + d_{M'}(D_2) + 4n - 8, \end{aligned}$$

contrary to $A \leq (4n - 3)(B - 2) + d_{M'}(D_1) + d_{M'}(D_2) - 6$. \square

Claim 3. $B = 0$.

Proof of Claim 3. Suppose on the contrary that $B \geq 1$. Then note that the number of vertices of degree 2 in $\partial D_1 \cap (\sigma_0 \cup \tau_0) - \{v'_0, v''_0\}$ is less than or equal to $d_{M'}(D_1) - 4$. Since every vertex in $\partial M' - \{v'_0, v''_0\}$ has degree 2 or at least 4 and since $4n - 2$ vertices of degree 2 do not occur consecutively on $\sigma_0 - \{v'_0, v''_0\}$ nor on $\tau_0 - \{v'_0, v''_0\}$, we have

$$A \leq (4n - 3)(B - 1) + d_{M'}(D_1) - 4.$$

On the other hand, by Lemma 4.12, we have

$$\begin{aligned} 4 &\leq \sum_{v \in \partial M'} (3 - d_{M'}(v)) + \sum_{D \in M'} (4 - d_{M'}(D)) \\ &\leq \sum_{v \in \partial M'} (3 - d_{M'}(v)) + \sum_{D \in M' - \{D_1\}} (4 - 4n) + (4 - d_{M'}(D_1)), \end{aligned}$$

so that

$$4 + \sum_{D \in M' - \{D_1\}} (4n - 4) \leq \sum_{v \in \partial M'} (3 - d_{M'}(v)) + (4 - d_{M'}(D_1)).$$

By Proposition 4.13(1),

$$B(4n - 4) \leq (F - 1)(4n - 4) = \sum_{D \in M' - \{D_1\}} (4n - 4);$$

hence

$$4 + B(4n - 4) \leq \sum_{v \in \partial M'} (3 - d_{M'}(v)) + (4 - d_{M'}(D_1)).$$

Here, since $d_{M'}(v'_0) = d_{M'}(v''_0) = 2$, we have $\sum_{v \in \partial M'} (3 - d_{M'}(v)) \leq A + 2 - B$, so that

$$4 + B(4n - 4) \leq (A + 2 - B) + (4 - d_{M'}(D_1)).$$

Thus

$$A \geq B(4n - 4) + B - 2 + d_{M'}(D_1) = (4n - 3)(B - 1) + d_{M'}(D_1) + 4n - 5,$$

contrary to $A \leq (4n - 3)(B - 1) + d_{M'}(D_1) - 4$. \square

By Claim 3, M consists of only one 2-cell, thus proving (2) and (3). \square

Corollary 4.17. *Let M be a reduced nontrivial annular diagram over $G(r; n) = \langle a, b \mid u_r^n \rangle$ satisfying the assumptions of Theorem 4.15. Then Figure 3 illustrates the only two possible shapes of M , in particular, the following hold.*

- (1) *If $\sigma \cap \tau \neq \emptyset$, then M consists of a single layer, and the number of faces of M is equal to the number of degree 4 vertices of M . Here the number of faces is variable.*

- (2) If $\sigma \cap \tau = \emptyset$, then M consists of two layers, and every vertex of M lies in ∂M . The number of faces of each of the layers is equal to the number of degree 4 vertices contained in σ , which in turn is equal to the the number of degree 4 vertices contained in τ . Here the number of faces per layer is variable.

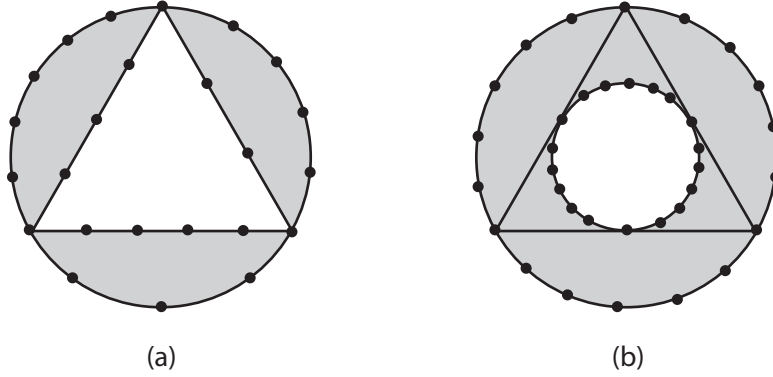


FIGURE 3. Two possible shapes of M when $n = 2$

The following notation will be used in Section 6.

Notation 4.18. Suppose that M is a connected annular map as in Figure 3, and let J be the outer boundary layer of M . Here we define the *outer boundary layer* of M to be the submap of M consisting of all 2-cells D such that the intersection of ∂D with the outer boundary of M contains an edge, together with the edges and vertices contained in ∂D . Choose a vertex, say v_0 , lying in both the outer and inner boundaries of J . Let α and δ be, respectively, the outer and inner boundary cycles of J starting from v_0 , where α is read clockwise and δ is read counterclockwise. Also let

$$\begin{aligned}\alpha &= e_{11}, e_{12}, \dots, e_{1s_1}, \dots, e_{t1}, e_{t2}, \dots, e_{ts_t}, \\ \delta^{-1} &= e'_{11}, e'_{12}, \dots, e'_{1k_1}, \dots, e'_{t1}, e'_{t2}, \dots, e'_{tk_t}\end{aligned}$$

be the decompositions into oriented edges in ∂J . Here t is the number of 2-cells in J , and for each $i = 1, \dots, t$, there is a 2-cell D_i of J such that $e_{i1}, e_{i2}, \dots, e_{is_i}, e'_{ik_i}, \dots, e'_{i2}, e'_{i1}$ are consecutive edges in a boundary cycle of D_i . We then denote the path $e_{i1}, e_{i2}, \dots, e_{is_i}$ by ∂D_i^+ and the path $e'_{i1}, e'_{i2}, \dots, e'_{ik_i}$ by ∂D_i^- .

5. TECHNICAL LEMMAS

In the remainder of this paper, suppose that p and n are integers greater than 1. Recall that the region, R , bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint $1/p$ forms a fundamental domain for the action of $\Gamma(1/p; n)$ on \mathbb{H}^2 (see Figure 1). Let $I_1(1/p; n)$ and $I_2(1/p; n)$ be the (closed or half-closed) intervals in \mathbb{R} defined as follows:

$$\begin{aligned} I_1(1/p; n) &= [0, r_1], \text{ where } r_1 = [p, 2n - 2], \\ I_2(1/p; n) &= [r_2, 1], \text{ where } r_2 = [p - 1, 2]. \end{aligned}$$

Then we may choose a fundamental domain R so that the intersection of \bar{R} with $\partial\mathbb{H}^2$ is equal to the union $\bar{I}_1(1/p; n) \cup \bar{I}_2(1/p; n) \cup \{\infty, 1/p\}$.

Lemma 5.1. *For any rational number $s \in I_1(1/p; n) \cup I_2(1/p; n)$, $CS(s)$ does not contain $((2n - 2)\langle p \rangle)$ as a subsequence.*

Proof. This is nothing other than [10, Proposition 5.1(1)]. \square

As an easy consequence of Lemmas 4.3 and 5.1, we obtain the following.

Corollary 5.2. *For any rational number $s \in I_1(1/p; n) \cup I_2(1/p; n)$, the cyclic word (u_s) cannot contain a subword w of the cyclic word $(u_{1/p}^{\pm n})$ which is a product of $4n - 1$ pieces but is not a product of less than $4n - 1$ pieces.*

If $\Gamma_{1/p}$ is the group of automorphisms of the Farey tessellation \mathcal{D} generated by reflections in the edges of \mathcal{D} with an endpoint $1/p$, and $\hat{\Gamma}_r$ is the group generated by $\Gamma_{1/p}$ and Γ_∞ , then the region, Q , bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$ on \mathbb{H}^2 . Let $I_1(1/p)$ and $I_2(1/p)$ be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of Q . Then the intervals $I_1(1/p)$ and $I_2(1/p)$ are given by $I_1(1/p) = \{0\}$ and $I_2(1/p) = [\frac{1}{p-1}, 1]$. Clearly $I_1(1/p) \subsetneq I_1(1/p; n)$ and $I_2(1/p) \subsetneq I_2(1/p; n)$. It was shown in [17, Proposition 4.6] that if two elements s and s' of $\hat{\mathbb{Q}}$ belong to the same $\hat{\Gamma}_{1/p}$ -orbit, then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(1/p)$.

Lemma 5.3. *For any rational number $s \in I_1(1/p) \cup I_2(1/p)$, if $s \neq 0$, then every term of $CS(s)$ is less than p .*

Proof. The assertion is nothing other than [11, Proposition 3.19]. \square

Lemma 5.4. *For any rational number $s \in I_1(1/p; n) \setminus I_1(1/p)$, $CS(s)$ contains a subsequence $(p + c, d\langle p \rangle, p + c')$ for some $c, c' \geq 1$ and $0 \leq d \leq 2n - 4$.*

Proof. Any rational number $s \in I_1(1/p; n) \setminus I_1(1/p)$, i.e., $0 < s < [p, 2n-2]$, has a continued fraction expansion $s = [l_1, \dots, l_t]$, where $t \geq 1$, $(l_1, \dots, l_t) \in (\mathbb{Z}_+)^t$ and $l_t \geq 2$, such that

- (i) $t \geq 3$, $l_1 = p$ and $l_2 = 1$; or
- (ii) $t \geq 2$, $l_1 = p$ and $2 \leq l_2 \leq 2n-3$; or
- (iii) $t \geq 1$ and $l_1 \geq p+1$.

If (i) happens, $CS(s)$ contains a subsequence $(p+1, p+1)$, so the assertion holds with $d = 0$. If (ii) happens, then $\tilde{s} = [l_2-1, l_3, \dots, l_t]$, where \tilde{s} denotes the rational number defined as in Lemma 3.8 for the rational number s so that $CS(\tilde{s}) = CT(s)$ and therefore $CT(s) = CS(\tilde{s})$ contains l_2-1 by Lemma 3.5. Hence $CS(s)$ contains a subsequence $(p+1, d\langle p \rangle, p+1)$ with $d = l_2-1$. Since $1 \leq d = l_2-1 \leq 2n-4$, the assertion holds. If (iii) happens, $CS(s)$ contains a subsequence $(p+c, p+c')$ for some $c, c' \geq 1$, so the assertion holds with $d = 0$. \square

Lemma 5.5. *For any rational number $s \in I_2(1/p; n) \setminus I_2(1/p)$, $CS(s)$ contains $(p-1, p, p-1)$ as a subsequence.*

Proof. Any rational number $s \in I_2(1/p; n) \setminus I_2(1/p)$, i.e., $[p-1, 2] \leq s < [p-1]$, has a continued fraction expansion $s = [l_1, \dots, l_t]$, where $t \geq 1$, $(l_1, \dots, l_t) \in (\mathbb{Z}_+)^t$ and $l_t \geq 2$, such that $t \geq 2$, $l_1 = p-1$ and $l_2 \geq 2$. Then by Lemma 3.5(2b), $CS(s)$ consists of $p-1$ and p without two consecutive terms (p, p) . It then follows that $CS(s)$ contains $(p-1, p, p-1)$ as a subsequence. \square

6. PROOF OF MAIN THEOREM 2.5(1)

Suppose on the contrary that there exist two distinct rational numbers s and s' in $I_1(1/p; n) \cup I_2(1/p; n)$ for which the simple loops α_s and $\alpha_{s'}$ are homotopic in $\mathbf{S}(1/p; n)$. Then u_s and $u_{s'}^{\pm 1}$ are conjugate in $G(1/p; n)$. By Lemma 4.11, there is a reduced nontrivial annular diagram M over $G(1/p; n) = \langle a, b \mid u_{1/p}^n \rangle$ with $(\phi(\alpha)) = (u_s)$ and $(\phi(\delta)) = (u_{s'}^{\pm 1})$, where α and δ are, respectively, outer and inner boundary cycles of M . Since $s, s' \in I_1(1/p; n) \cup I_2(1/p; n)$, we see by Lemma 5.1 that $CS(\phi(\alpha))$ and $CS(\phi(\delta))$ do not contain $((2n-2)\langle p \rangle)$ as a subsequence. So by Corollary 4.17, M is shaped as in Figure 3(a) or Figure 3(b).

Lemma 6.1. *M is shaped as in Figure 3(a), that is, M satisfies the conclusion Corollary 4.17(1).*

Proof. Suppose on the contrary that M is shaped as in Figure 3(b). Then $(\phi(\alpha)) = (u_s)$ contains a subword of the cyclic word $(u_{1/p}^{\pm n})$ which is a product of $4n - 2$ pieces but is not a product of less than $4n - 2$ pieces (see Convention 4.7(3) and Theorem 4.15(4)). Since $4n - 2 \geq 6$, this together with Lemma 4.2(1c) implies that $CS(\phi(\alpha)) = CS(s)$ contains a term p . Furthermore, we have $s \neq 0$, because $CS(u_0) = ((2))$ by Remark 3.2. Then by Lemma 5.3, $s \notin I_1(1/p) \cup I_2(1/p)$. By Lemmas 5.4 and 5.5, the cyclic word (u_s) contains a subword w for which $S(w)$ is a subsequence of $CS(s)$ such that

$$S(w) = \begin{cases} (p + c, d\langle p \rangle, p + c') & \text{if } s \in I_1(1/p; n) \setminus I_1(1/p); \\ (p - 1, p, p - 1) & \text{if } s \in I_2(1/p; n) \setminus I_2(1/p), \end{cases}$$

where $c, c' \geq 1$ and $0 \leq d \leq 2n - 4$.

Claim. *There is a face D in the outer boundary layer of M such that $\phi(\partial D^+)$ is a subword of w (recall Notation 4.18).*

Proof of Claim. Suppose that there is no such face. Then either (i) there is a face, D' , in the outer boundary layer of M such that $\phi(\partial D'^+) \equiv u w v$ for some words u and v such that at least one of them is nonempty, or (ii) there are two successive faces, say D_1 and D_2 , in the outer boundary layer of M such that $\phi(\partial D_1^+) \equiv u w_1$ and $\phi(\partial D_2^+) \equiv w_2 v$ where u, v, w_1 and w_2 are nonempty words such that $w \equiv w_1 w_2$. Since $S(w)$ is a subsequence of $CS(s)$, this implies that the first or the last component of $S(w)$ is also a component of $CS(\phi(\partial D')) = ((2n\langle p \rangle))$, a contradiction. \square

For such a face D as in the statement of the above claim, since $CS(\phi(\partial D)) = ((2n\langle p \rangle))$, this yields that $S(\phi(\partial D^-))$ must contain (p, p) or $(\ell, (2n - 3)\langle p \rangle, \ell')$ as a subsequence for some $\ell, \ell' \in \mathbb{Z}_+$. But then by Lemma 4.2(1), the word $\phi(\partial D^-)$ cannot be expressed as a product of 2 pieces of $(u_r^{\pm 1})$, contradicting Figure 3(b) (cf. Corollary 4.17(2)). \square

Lemma 6.2. *For every face D in M , $S(\phi(\partial D^\pm))$ contains a term p .*

Proof. Suppose that $S(\phi(\partial D^+))$ does not contain a term p . Then $S(\phi(\partial D^+))$ is of the form either (ℓ) with $1 \leq \ell \leq p - 1$ or (ℓ_1, ℓ_2) with $1 \leq \ell_1, \ell_2 \leq p - 1$. In the first case, $\phi(\partial D^-)$ is a product of $4n - 1$ pieces, but is not a product of less than $4n - 1$ pieces. But since $\phi(\partial D^-)$ is a subword of $(\phi(\delta)) = (u_{s'})$, this gives a contradiction to Corollary 5.2. In the second case, $S(\phi(\partial D^-)) = (p - \ell_1, (2n - 2)\langle p \rangle, p - \ell_2)$, which implies that $CS(\phi(\delta)) = CS(s')$ contains $((2n - 2)\langle p \rangle)$ as a subsequence, contrary to Lemma 5.1.

The same argument applies to $S(\phi(\partial D^-))$. \square

Lemma 6.3. $s, s' \notin I_1(1/p) \cup I_2(1/p)$.

Proof. By Lemma 6.2, both $CS(s)$ and $CS(s')$ contain a term bigger than or equal to p . Clearly $s, s' \neq 0$, since $CS(u_0) = ((2))$ by Remark 3.2. Then the assertion follows from Lemma 5.3. \square

Lemma 6.4. $s, s' \notin I_2(1/p; n)$.

Proof. Suppose on the contrary that s or s' is contained in $I_2(1/p; n)$. Without loss of generality, assume that $s \in I_2(1/p; n)$. Then by Lemma 6.3, $s \in I_2(1/p; n) \setminus I_2(1/p)$. By Lemma 5.5, $CS(\phi(\alpha)) = CS(s)$ contains $(p-1, p, p-1)$ as a subsequence without two consecutive terms (p, p) . Let w to be the subword of (u_s) such that $S(w) = (p-1, p, p-1)$. Then arguing as in the proof of Lemma 6.1, we see that there is a face D such that $\phi(\partial D^+)$ is a subword of w , so that $S(\phi(\partial D^-))$ contains $(\ell, (2n-3)\langle p \rangle, \ell')$ as a subsequence for some $\ell, \ell' \in \mathbb{Z}_+$. This implies that $CS(\phi(\delta)) = CS(s')$ contains $(p+1, d\langle p \rangle, p+1)$ as a subsequence for some $d \geq 2n-3$. Let $s' = [a_1, a_2, \dots, a_t]$ be a continued fraction expansion. Then since $CS(s')$ consists of p and $p+1$, we see $a_1 = p$ by Lemma 3.5. If $a_2 \geq 2$, then since $CT(s')$ contains d as a component, we see by using Lemmas 3.5 and 3.8 that $a_2 - 1 = d$ or both $a_2 - 1 = d - 1$ and $t \geq 3$, i.e., $a_2 = d + 1$ or both $a_2 = d$ and $t \geq 3$. Since $s \in I_1(r; n) \cup I_2(r; n)$, we must have $a_2 \leq 2n-3$, and therefore $a_2 = d = 2n-3$ and $t \geq 3$. Also if $a_2 = 1$, then by Lemma 3.5, $CS(s')$ does not contain (p, p) as a subsequence, so we must have $d = 1$, so, $a_2 = d = 2n-3$. In this case, clearly $t \geq 3$. Thus in either case, $s' = [p, 2n-3, a_3, \dots, a_t]$ with $t \geq 3$, i.e., $[p, 2n-3] < s' < [p, 2n-2]$. Thus, by Lemmas 3.5 and 3.8, $CS(s')$ contains $(p+1, (2n-4)\langle p \rangle, p+1)$ too as a subsequence. Arguing similarly as in the the proof of Lemma 6.1, we see that there is a face D such that $\phi(\partial D^-)$ is a subword of the word corresponding to the subsequence $(p+1, (2n-4)\langle p \rangle, p+1)$. Since $CS(\phi(\partial D)) = ((2n\langle p \rangle))$, this yields that $S(\phi(\partial D^+))$ must contain (p, p) as a subsequence, a contradiction. \square

Lemma 6.5. Every term of $CS(s)$ and $CS(s')$ is greater than or equal to p .

Proof. By Lemmas 6.3 and 6.4, $s, s' \in I_1(1/p; n) \setminus I_1(1/p)$. Then by Lemma 5.4, $CS(s)$ and $CS(s')$ contain a term greater than p . So by Lemma 3.5, every term of $CS(s)$ and $CS(s')$ is greater than or equal to p . \square

Lemma 6.6. Neither $CS(s)$ nor $CS(s')$ can contain a term of the form $p+d$ with $1 \leq d \leq p-1$.

Proof. Suppose on the contrary that $CS(\phi(\alpha)) = CS(s)$ or $CS(\phi(\delta)) = CS(s')$, say $CS(s)$, contains a term of the form $p+d$ with $1 \leq d \leq p-1$. Then there

exist two 2-cells D_1 and D_2 in M as illustrated in Figure 4(a) which follows Convention 6.7 below such that

- (i) $\partial D_1^+ \partial D_2^+$ is a subpath of an outer boundary cycle of M ;
- (ii) $S(\phi(\partial D_1^+)) = (\dots, \ell_1)$, where $1 \leq \ell_1 \leq p$;
- (iii) $S(\phi(\partial D_2^+)) = (\ell_2, \dots)$, where $1 \leq \ell_2 \leq p$; and
- (iv) $S(\phi(\partial D_1^+ \partial D_2^+)) = (\dots, \ell_1 + \ell_2, \dots)$, where $\ell_1 + \ell_2 = p + d$.

Since $p + d < 2p$, $\ell_1 < p$ or $\ell_2 < p$. Here, if $\ell_1 < p$ and $\ell_2 < p$, then since both $S(\phi(\partial D_1^-))$ and $S(\phi(\partial D_2^-))$ contain a term p by Lemma 6.2, we see that $S(\phi(\partial D_1^- \partial D_2^-))$ contains a subsequence $(p, 2p - (\ell_1 + \ell_2), p)$ as shown in Figure 4(b). So $CS(\phi(\delta)) = CS(s')$ contains a term $2p - (\ell_1 + \ell_2) = p - d < p$, which contradicts Lemma 6.5. On the other hand, if $\ell_1 < p$ and $\ell_2 = p$, then $S(\phi(\partial D_1^- \partial D_2^-))$ contains a subsequence $(p, p - \ell_1, p)$ as shown in Figure 4(c). So $CS(\phi(\delta)) = CS(s')$ contains a term $p - \ell_1 < p$, again a contradiction to Lemma 6.5. Obviously a similar contradiction is obtained if $\ell_1 = p$ and $\ell_2 < p$ as shown in Figure 4(d). \square

Convention 6.7. Recall that M is shaped as in Figure 3(a). In Figures 4 and 5, the upper complementary region is regarded as the unbounded region of $\mathbb{R}^2 - M$. Thus an outer boundary cycle runs the upper boundary from left to right. Also the change of directions of consecutive arrowheads represents the change from positive (negative, resp.) words to negative (positive, resp.) words, and a dot represents a vertex whose position is clearly identified. Furthermore a number such as ℓ_1, ℓ_2, p , etc represents the length of the corresponding positive (or negative) word.

Lemma 6.8. *Neither $CS(s)$ nor $CS(s')$ can contain a term greater than $2p$.*

Proof. Suppose on the contrary that $CS(\phi(\alpha)) = CS(s)$ or $CS(\phi(\delta)) = CS(s')$, say $CS(s)$, contains a term greater than $2p$. Then by Lemma 6.2, there exist three 2-cells D_1, D_2 and D_3 in M as illustrated in Figure 5(a) which follows Convention 6.7 such that

- (i) $\partial D_1^+ \partial D_2^+ \partial D_3^+$ is a subpath of an outer boundary cycle of M ;
- (ii) $S(\phi(\partial D_1^+)) = (\dots, \ell_1)$, where $1 \leq \ell_1 \leq p$;
- (iii) $S(\phi(\partial D_2^+)) = (p)$;
- (iv) $S(\phi(\partial D_3^+)) = (\ell_2, \dots)$, where $1 \leq \ell_2 \leq p$; and
- (v) $S(\phi(\partial D_1^+ \partial D_2^+ \partial D_3^+)) = (\dots, \ell_1 + p + \ell_2, \dots)$, where $\ell_1 + p + \ell_2 > 2p$.

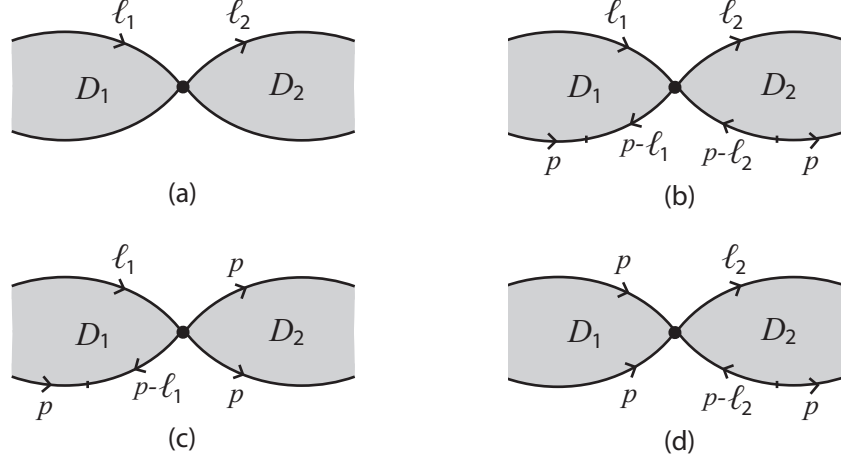


FIGURE 4. Lemma 6.6

Here, if $\ell_1 < p$, then since $S(\phi(\partial D_1^-))$ contains a term p by Lemma 6.2, and since $S(\phi(\partial D_2^-)) = ((2n-1)\langle p \rangle)$, we see that $S(\phi(\partial D_1^- \partial D_2^-))$ contains a subsequence $(p, p - \ell_1, p)$ as shown in Figure 5(b). So $CS(\phi(\delta)) = CS(s')$ contains a term $p - \ell_1 < p$, which contradicts Lemma 6.5. On the other hand, if $\ell_1 = p$, then $S(\phi(\partial D_1^- \partial D_2^-))$ contains a subsequence $(2p, p, p)$ as shown in Figure 5(c). So $CS(\phi(\delta)) = CS(s')$ contains both a term p and a term $2p + \ell$ with $\ell \geq 0$. But since $2p + \ell > p + 1$, we obtain a contradiction to Lemma 3.5. \square

By Lemmas 6.5, 6.6 and 6.8, the only possibility is that $CS(s) = CS(s') = ((2p, 2p))$, but this is an obvious contradiction, because s and s' are distinct.

The proof of Main Theorem 2.5(1) is now completed. \square

7. PROOF OF MAIN THEOREM 2.5(2)

Suppose on the contrary that there exists a rational number s in $I_1(1/p; n) \cup I_2(1/p; n)$ for which the simple loop α_s is peripheral in $\mathbf{S}(1/p; n)$. Then u_s is conjugate to a^t or b^t in $G(1/p; n)$ for some integer $t \geq 1$. Suppose that u_s is conjugate to a^t in $G(1/p; n)$ (the case when u_s is conjugate to b^t in $G(1/p; n)$ is treated similarly). By Lemma 4.11, there is a reduced nontrivial annular diagram M over $G(1/p; n) = \langle a, b \mid u_{1/p}^n \rangle$ with $(\phi(\alpha)) = (u_s)$ and $(\phi(\delta)) = (a^{\pm t})$, where α and δ are, respectively, outer and inner boundary cycles of M . Furthermore, by Corollary 4.8, M is a $[4, 4n]$ -map.

Let the outer and inner boundaries of M be denoted by σ and τ , respectively.

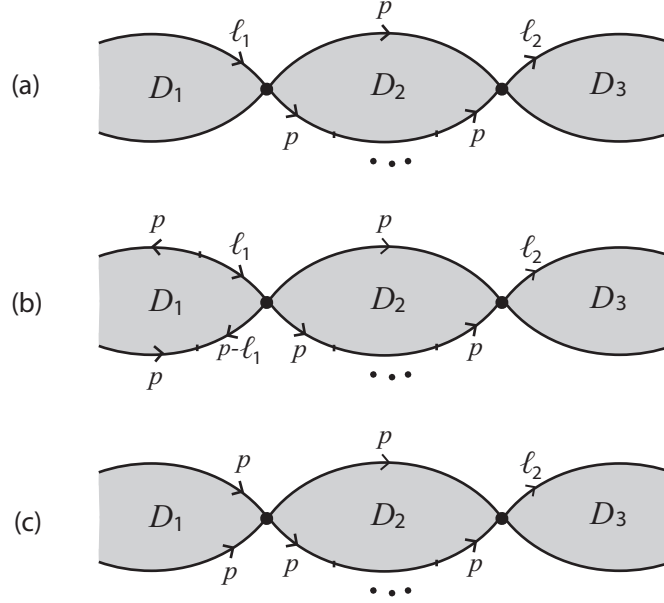


FIGURE 5. Lemma 6.8

Lemma 7.1. *The outer and inner boundaries σ and τ are simple.*

Proof. If σ is not simple, then the same proof of Lemma 4.16 yields a contradiction. So suppose that τ is not simple. Then there is an extremal disk, say J , such that J is properly contained in M with $\partial J \subset \tau$ and connected to the rest of M by a single vertex. Then J is a connected and simply-connected $[4, 4n]$ -map. Since $(u_{1/p}^n)$ is alternating but $(\phi(\delta)) = (a^{\pm t})$ is not, there is no vertex in ∂J with $d_J(v) = 2$. But this is a contradiction to Proposition 4.13(2) applied to J . \square

Lemma 7.2. *There is no edge contained in $\sigma \cap \tau$.*

Proof. Suppose on the contrary that $\sigma \cap \tau$ contains an edge. As illustrated in Figure 2 in Section 4, there is a submap J of M such that

- (i) J is bounded by a simple closed path of the form $\sigma_1 \tau_1$, where $\sigma_1 \subseteq \sigma$ and $\tau_1 \subseteq \tau$;
- (ii) J is connected to the rest of M by two distinct vertices, say v_1 and v_2 , where $\sigma_1 \cap \tau_1 = \{v_1, v_2\}$ and v_1 is an endpoint of an edge contained in $\sigma \cap \tau$. Note that $d_J(v_1) = d_M(v_1) - 1 \geq 2$ and $d_J(v_2) \geq 2$.

Then J is a connected and simply connected $[4, 4n]$ -map. Since $(u_{1/p}^n)$ is alternating but $(\phi(\delta)) = (a^{\pm t})$ is not, there is no vertex in $\tau_1 - \{v_1, v_2\}$ with

$d_J(v) = 2$. Also, since both $(u_{1/p}^n)$ and $(\phi(\alpha)) = (u_s)$ are alternating, there is no vertex in $\sigma_1 - \{v_1, v_2\}$ with $d_J(v) = 3$. So we put

- A = the number of vertices v in $\sigma_1 - \{v_1, v_2\}$ such that $d_J(v) = 2$;
- B_1 = the number of vertices v in $\sigma_1 - \{v_1, v_2\}$ such that $d_J(v) \geq 4$;
- B_2 = the number of vertices v in $\tau_1 - \{v_1, v_2\}$ such that $d_J(v) \geq 4$;
- C = the number of vertices v in $\tau_1 - \{v_1, v_2\}$ such that $d_J(v) = 3$.

Since $s \in I_1(1/p; n) \cup I_2(1/p; n)$, we see by Corollary 5.2 together with Convention 4.7(3) that $4n - 2$ degree 2 vertices cannot occur consecutively on $\sigma_1 - \{v_1, v_2\}$, so that $A \leq (4n - 3)(B_1 + 1)$. We will derive a contradiction to this inequality using Proposition 4.13(2) applied to J .

First if $d_J(v_1) = 2$ and $d_J(v_2) = 2$, then

$$\begin{aligned} A + 2 &\geq (4n - 3)(B_1 + B_2) + (4n - 4)\lceil C/2 \rceil + 4n \\ &\geq (4n - 3)B_1 + 4n = (4n - 3)(B_1 + 1) + 3, \end{aligned}$$

so that $A \geq (4n - 3)(B_1 + 1) + 1$, contrary to $A \leq (4n - 3)(B_1 + 1)$. Second either if $d_J(v_1) = 2$ and $d_J(v_2) = 3$ or if $d_J(v_1) = 3$ and $d_J(v_2) = 2$, then

$$\begin{aligned} A + 1 &\geq (4n - 3)(B_1 + B_2) + (4n - 4)\lceil (C + 1)/2 \rceil + 4n \\ &\geq (4n - 3)B_1 + 4n = (4n - 3)(B_1 + 1) + 3, \end{aligned}$$

so that $A \geq (4n - 3)(B_1 + 1) + 2$, contrary to $A \leq (4n - 3)(B_1 + 1)$. Third either if $d_J(v_1) = 2$ and $d_J(v_2) \geq 4$ or if $d_J(v_1) \geq 4$ and $d_J(v_2) = 2$, then

$$\begin{aligned} A + 1 &\geq (4n - 3)(B_1 + B_2 + 1) + (4n - 4)\lceil C/2 \rceil + 4n \\ &\geq (4n - 3)(B_1 + 1) + 4n, \end{aligned}$$

so that $A \geq (4n - 3)(B_1 + 1) + 4n - 1$, contrary to $A \leq (4n - 3)(B_1 + 1)$. Fourth if $d_J(v_1) = 3$ and $d_J(v_2) = 3$, then

$$\begin{aligned} A &\geq (4n - 3)(B_1 + B_2) + (4n - 4)\lceil (C + 2)/2 \rceil + 4n \\ &\geq (4n - 3)B_1 + 4n = (4n - 3)(B_1 + 1) + 3, \end{aligned}$$

contrary to $A \leq (4n - 3)(B_1 + 1)$. Fifth either if $d_J(v_1) = 3$ and $d_J(v_2) \geq 4$ or if $d_J(v_1) \geq 4$ and $d_J(v_2) = 3$, then

$$\begin{aligned} A &\geq (4n - 3)(B_1 + B_2 + 1) + (4n - 4)\lceil (C + 1)/2 \rceil + 4n \\ &= (4n - 3)(B_1 + 1) + 4n, \end{aligned}$$

contrary to $A \leq (4n - 3)(B_1 + 1)$. Finally if $d_J(v_1) \geq 4$ and $d_J(v_2) \geq 4$, then

$$\begin{aligned} A &\geq (4n - 3)(B_1 + B_2 + 2) + (4n - 4)\lceil C/2 \rceil + 4n \\ &= (4n - 3)(B_1 + 2) + 4n, \end{aligned}$$

contrary to $A \leq (4n - 3)(B_1 + 1)$. \square

At this point, we newly let A , B and C be the numbers defined for M as in Proposition 4.13. Then by Proposition 4.13(2) applied to M , we have $A \geq (4n - 3)B + (4n - 4)\lceil C/2 \rceil$. Furthermore, we newly put

$A_1 =$ the number of vertices v in σ such that $d_M(v) = 2$;

$B_1 =$ the number of vertices v in σ such that $d_M(v) \geq 4$.

Claim. $A_1 > (4n - 3)B_1$.

Proof of Claim. First suppose that $\sigma \cap \tau = \emptyset$. Since $(u_{1/p}^n)$ is alternating but $(\phi(\delta)) = (a^{\pm t})$ is not, there is no vertex in τ with degree 2, and therefore $A = A_1$. Clearly there is at least one vertex in τ , say v . Since $d_M(v) \geq 3$, we have $(4n - 3)B + (4n - 4)\lceil C/2 \rceil > (4n - 3)B_1$. Here, since $A = A_1$, we have $A_1 > (4n - 3)B_1$, as desired.

Next suppose that $\sigma \cap \tau \neq \emptyset$. By Lemma 7.2, $\sigma \cap \tau$ consists of finitely many vertices in M . Then for every $v \in \sigma \cap \tau$, clearly $d_M(v) \geq 4$. Furthermore, since $(\phi(\alpha)) = (u_s)$ is alternating while $(\phi(\delta)) = (a^{\pm t})$ is not, we see that $d_M(v)$ is an odd number. Then $d_M(v) \geq 4$ is an odd number, which implies that $d_M(v) \geq 5$ and therefore M does not satisfy the condition in Remark 4.14(2). Hence $A > (4n - 3)B + (4n - 4)\lceil C/2 \rceil$ by the remark. Clearly $(4n - 3)B + (4n - 4)\lceil C/2 \rceil \geq (4n - 3)B_1$. Here, since $A = A_1$ by reasoning as above, we have $A_1 > (4n - 3)B_1$, as desired. \square

The above claim implies that σ contains $4n - 2$ consecutive degree 2 vertices. But then the cyclic word $(\phi(\alpha)) = (u_s)$ contains a subword w of $(u_{1/p}^{\pm n})$ which is a product of $4n - 1$ pieces but is not a product of less than $4n - 1$ pieces. This contradiction to Corollary 5.2 completes the proof of Main Theorem 2.5(2). \square

8. PROOF OF MAIN THEOREM 2.5(3)

We recall the following lemma.

Lemma 8.1. (1) ([6, Theorem 6.3]) *Suppose that v is a cyclically alternating word which represents the trivial element in $G(K(1/p)) = \langle a, b \mid u_{1/p} \rangle$. Then the cyclic word (v) contains a subword w of the cyclic word $(u_{1/p}^{\pm 1})$ such that $S(w)$ is (p, ℓ) or (ℓ, p) for some $\ell \in \mathbb{Z}_+$.*

(2) ([10, Corollary 4.12]) Suppose that v is a cyclically alternating word which represents the trivial element in $G(1/p; n) = \langle a, b \mid u_{1/p}^n \rangle$. Then the cyclic word (v) contains a subword w of the cyclic word $(u_{1/p}^{\pm n})$ such that $S(w) = (\ell, (2n-2)\langle p \rangle, \ell')$, where $\ell, \ell' \in \mathbb{Z}_+$.

Suppose on the contrary that there is a rational number s in $I_1(1/p; n) \cup I_2(1/p; n)$ for which $u_s^t = 1$ in $G(1/p; n)$ for some integer $t \geq 1$. Then clearly $u_s^t = 1$ also in $G(K(1/p))$. Since $G(K(1/p))$ is torsion-free, $u_s = 1$ in $G(K(1/p))$. By [6, Main Theorem 2.3], this implies that s lies in the $\hat{\Gamma}_{1/p}$ -orbit of $1/p$ or ∞ . Hence $|u_s| > |u_{1/p}|$.

On the other hand, since $u_s^t = 1$ in $G(1/p; n)$, Lemma 8.1(2) implies that we may write $\bar{u}_s^t \equiv wz$, where \bar{u}_s is a cyclic permutation of u_s and w is a subword of (u_s^t) as described in Lemma 8.1(2). Since $s \in I_1(1/p; n) \cup I_2(1/p; n)$, w cannot be contained in \bar{u}_s by Lemma 5.1. This yields that \bar{u}_s is a proper initial subword of w . Then \bar{u}_s is a subword of the cyclic word $(u_{1/p}^{\pm n})$. Here, since $|\bar{u}_s| = |u_s| > |u_{1/p}|$, we may put $\bar{u}_s \equiv v^d v_1$, where $d \in \mathbb{Z}_+$, v is a cyclic permutation of $u_{1/p}$ or $u_{1/p}^{-1}$ and $|v_1| < |u_{1/p}|$. Note that $|v_1| \geq 1$, for otherwise we would have $\bar{u}_s \equiv v^d$ so that $CS(s) = ((2d\langle p \rangle))$, which yields that $d \geq 2$ since $s \neq 1/p$ and that $s = d/dp$ by Remark 3.11, a contradiction.

Then $v_1 = \bar{u}_s = 1$ in $G(K(1/p))$. Moreover v_1 is a proper beginning subword of v and so a proper beginning subword of \bar{u}_s . This implies that v_1 is cyclically alternating, because \bar{u}_s is cyclically alternating and $|v_1| = |u_s| - d|u_{1/p}|$ is even. Also since $v_1 = 1$ in $G(K(1/p))$, Lemma 8.1(1) implies that $|v_1| > \frac{1}{2}|u_{1/p}| = \frac{1}{2}|v|$. Now, let v_2 be the terminal subword of v such that $v \equiv v_1 v_2$. Then v_2 is also cyclically alternating and $v_2 = 1$ in $G(K(1/p))$. Thus the above argument implies that $|v_2| > \frac{1}{2}|v|$. This contradicts the inequality $|v_2| = |v| - |v_1| < \frac{1}{2}|v|$. \square

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